

On the Solvability of 2-pair Unicast Networks — A Cut-based Characterization

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Abstract

In this paper, we propose a subnetwork decomposition/combination approach to investigate the single rate 2-pair unicast problem. It is shown that the solvability of a 2-pair unicast problem is completely determined by four specific link subsets, namely, $\mathcal{A}_{1,1}$, $\mathcal{A}_{2,2}$, $\mathcal{A}_{1,2}$ and $\mathcal{A}_{2,1}$ of its underlying network. As a result, an efficient cut-based algorithm to determine the solvability of a 2-pair unicast problem is presented.

Index Terms

Network coding, Capacity, 2-pair unicast problem, \mathcal{A} -set.

I. INTRODUCTION

It is an important issue to decide the admissible rate region for a multi-source multi-sink communication network in network information theory. The history of the research can be traced back to the earlier work of Elias *et al.* [1], as well as Ford and Fulkerson [2] in 1956. It was shown that the capacity of every one-source one-sink (point-to-point) network can be characterized by its minimum cut (Max-flow Min-cut Theorem). In [3]-[5], Yeung and Zhang presented the inner and outer bounds of the admissible rate region for a distributed source coding system. Based on these works, Ahlswede *et al.* [6] showed that the Max-flow Min-cut capacity can be achieved for multicast networks by using a coding strategy in their seminal work on network coding. Later on, Li *et al.* [7] proved that linear network coding is sufficient to achieve the Max-flow Min-cut capacity for multicast networks.

Unlike the one source networks, for a general multi-source multi-sink network with arbitrary transmission requirements, the Max-flow Min-cut capacity bound can be quite loose. Although some outer and inner bounds [8]-[12], and an entropy characterization [13] have been proposed, the explicit evaluation of the rate region for a general multi-source multi-sink network is very challenging. So many previous studies concentrated on the k -pair networks.

The k -pair communication problem, which is also known as the multiple unicast sessions, aims at supporting k independent point-to-point communications. Without network coding, i.e., just using pure routing strategy, it is the conventional multi-commodity flow (MCF) problem. For the MCF problem, a fractional achievable rate can be found using linear programming, but it is generally NP-hard to find an integral solution, except for the directed acyclic case, for which there is a polynomial algorithm of using the pebbling game [14], which is of

extraordinary complexity. When considering network coding, it is conjectured that there is no more advantage than using fractional routing in undirected networks. This is known as the *undirected k -pair conjecture* [15] and has been verified just for a few classes (see [8], [15] and [16]). In contrast, network coding can provide a significant rate increase in directed k -pair networks [15]. Except for the undirected 2-pair networks (and a few other families, see [15] for reference), whose capacity regions can be characterized by the cut condition, it is very difficult to evaluate the exact rate region for a k -pair network in general.

In this paper, we propose a subnetwork decomposition/combination approach to investigate the underlying graph structure of the *directed acyclic 2-pair unicast networks*. Our result shows that the solvability of a 2-pair unicast problem is completely determined by four particular link subsets of the underlying network, namely, $\mathcal{A}_{1,1}$, $\mathcal{A}_{2,2}$, $\mathcal{A}_{1,2}$ and $\mathcal{A}_{2,1}$, which can be considered as the most “important” links of the 2-pair network. As a result, we show that a 2-pair unicast problem is solvable if and only if the underlying network contains a copy of one of the four networks shown in Fig.1. Consequently, an efficient cut-based algorithm to determine the solvability of a 2-pair unicast problem is presented. In addition, a new proof that nonlinear network coding is unnecessary for a 2-pair unicast problem is obtained ¹.

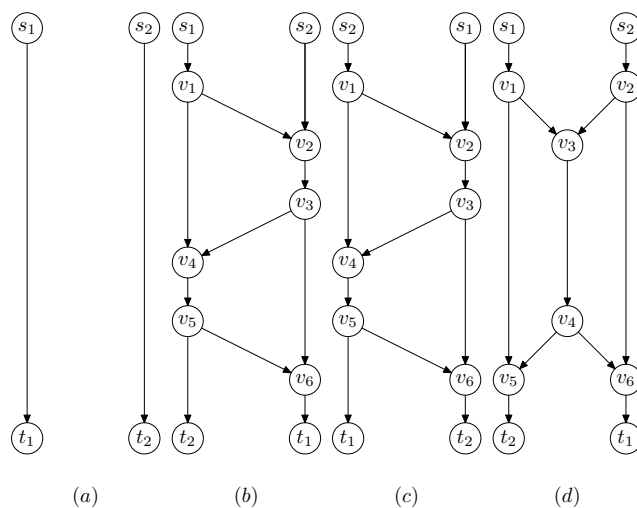


Fig. 1: Four underlying networks of 2-pair unicast networks.

Our method is based on the following two steps: Firstly, decompose a n -source m -sink network into nm point-to-point subnetworks (for the 2-pair network, $n = m = 2$). Since the properties of a point-to-point network can be easily inferred, this step simplify the initial multi-source multi-sink network coding problem. Secondly, consider the cut set relations of these point-to-point subnetworks. Such relations are shown to contain valuable information of the whole network structure. The first step can simplify the initial problem and the second step can yield a global picture of the original network. A number of “path operations” are used in this paper. That is, a desired path is usually constructed by joining a number of path sections, and conversely, a path will be decomposed into different sections according to particular demands. Our method finally provides an efficient cut-based algorithm to determine the solvability of the 2-pair unicast problem.

The rest of the paper is organized as follows. In Section II, some notations and results which will be used in

¹When we finished the first version of this paper, another independent work by Chih-Chun Wang and Ness B. Shroff [18] was published in the ISIT 2007 proceedings. They also derived the four configurations of Fig.1 and presented another characterization as well as a polynomial algorithm of using pebbling games to determine the solvability of 2-pair unicast networks.

the sequel are given. The underlying structure of the 2-pair network is presented in Section III. The solvability of the 2-pair unicast problem is analyzed in Section IV. The paper is concluded in Section V.

II. PRELIMINARIES

In this paper, most of the discussions are from a graph theoretical point of view. As a preparation, we introduce some basic definitions as well as some simple but frequently used results in this section.

A. Communication Network, Minimum Cut, and \mathcal{A} -Set

A *communication network* $\mathcal{N} = (V, E, S, T)$ consists of a directed acyclic graph (DAG) $G = (V, E)$, a source node set $S \subseteq V$, a sink node set $T \subseteq V$, and a nonnegative capacity $c(e)$ for each link $e \in E$. When $S = \{s\}$ and $T = \{t\}$, i.e., the network has a single source node and a single sink node, it is called a *point-to-point network* and denoted by (V, E, s, t) . Given $s_i \in S$ and $t_j \in T$, it yields a point-to-point network $\mathcal{N}_{i,j} = (V, E, s_i, t_j)$ by considering the other source and sink nodes as internal nodes. Thus there are totally $|S| \times |T|$ point-to-point networks underlying the network $\mathcal{N} = (V, E, S, T)$.

Let $\mathcal{N} = (V, E, s, t)$ be a point-to-point network and let $V = A \cup \bar{A}$ be a vertex partition of $G = (V, E)$ such that $s \in A$ and $t \in \bar{A} = V \setminus A$. An *s-t cut* C is a collection of all the edges from A to \bar{A} . The capacity of C is defined as $\sum_{e \in C} c(e)$. The minimum of the cut capacities for all *s-t* cuts is called the *minimum cut capacity* and denoted by $C_{\mathcal{N}}(s, t)$ or $C(s, t)$ when there is no ambiguity. A *minimum cut* is a cut with the minimum cut capacity. Noticing that there may be a number of minimum cuts within a point-to-point network, the union of those minimum cuts is called the *\mathcal{A} -set* (or the *cut set*) of the network (see [20]). Note that the \mathcal{A} -set plays an important role in this work.

In this paper, the edges of the network are assumed to have unit capacity, i.e., $c(e) = 1$. In this case, the well-known Max-flow Min-cut Theorem indicates that the *maximum flow* f , i.e., the number of edge-disjoint paths from s to t equals to $C(s, t)$, the minimum cut capacity. We call a family of k ($k \in \mathbb{N}$) edge-disjoint paths with common source and sink nodes as an *edge-disjoint k-path*, and denote it by $P^{(k)}$. For a point-to-point network (V, E, s, t) with the maximum flow f , it may generally have a number of edge-disjoint f -paths from s to t . Those edge-disjoint f -paths will be denoted by $P_1^{(f)}$, $P_2^{(f)}$, and so on.

Proposition 2.1: Let $\mathcal{N} = (V, E, s, t)$ be a point-to-point network with maximum flow f . Let $P_1^{(f)}$, $P_2^{(f)}$, \dots , and $P_k^{(f)}$ be all the edge-disjoint f -paths from s to t . Then we have

$$\mathcal{A} = \bigcap_{i=1}^k P_i^{(f)},$$

where \mathcal{A} is the \mathcal{A} -set of \mathcal{N} and $P_i^{(f)}$ is considered as the collection of its edges.

Proof: Let $e \in \mathcal{A}$. Then there exist a minimum cut $C = \{e_1, e_2, \dots, e_f\}$ such that $e \in C$. Let $V = A \cup \bar{A}$ be the vertex partition corresponding to C . Since C consists of all the edges from A to \bar{A} , each path of $P_i^{(f)}$ intersects C for any $i = 1, 2, \dots, k$. The edge-disjoint condition yields $|P_i^{(f)} \cap C| = f$. Since $|C| = f$, we have $C \subset P_i^{(f)}$ for $i = 1, 2, \dots, k$, and thus $e \in \bigcap_{i=1}^k P_i^{(f)}$. Therefore $\mathcal{A} \subseteq \bigcap_{i=1}^k P_i^{(f)}$.

On the other hand, let $e \in \bigcap_{i=1}^k P_i^{(f)}$ and consider $\mathcal{N}' = \mathcal{N} \setminus \{e\}$, the network deduced by deleting e from \mathcal{N} . We declare that $C_{\mathcal{N}'}(s, t) = f - 1$. In fact, if $C_{\mathcal{N}'}(s, t) = f$, then there will be an edge-disjoint f -path

from s to t which does not pass through e , which contradicts to the assumption $e \in \bigcap_{i=1}^k P_i^{(f)}$. Also, $C_{\mathcal{N}'}(s, t)$ can not be less than $f - 1$ since \mathcal{N}' is formed by deleting just one edge from \mathcal{N} . Now take a minimum cut $C' = \{e'_1, e'_1, \dots, e'_{f-1}\}$ of \mathcal{N}' and let $V = B \cup \bar{B}$ be the vertex partition corresponding to C' . Consider the tail and the head of the edge e , denoted by $\text{tail}(e)$ and $\text{head}(e)$, respectively. If both of them are in B , or both are in \bar{B} , or $\text{tail}(e) \in \bar{B}$ and $\text{head}(e) \in B$, then C' also yields a cut of \mathcal{N} , which contradicts to that $C_{\mathcal{N}}(s, t) = f$. Thus $\text{tail}(e) \in B$ and $\text{head}(e) \in \bar{B}$, which implies that $\{e\} \cup C'$ is a (minimum) cut of \mathcal{N} . Hence $e \in \mathcal{A}$ which gives $\bigcap_{i=1}^k P_i^{(f)} \subseteq \mathcal{A}$. \blacksquare

Obviously, a 2-source 2-sink network yields four point-to-point networks $\mathcal{N}_{i,j} = (V, E, s_i, t_j)$, for $i, j = 1, 2$. In the following part, we use $\mathcal{A}_{i,j}$ to denote the \mathcal{A} -set of $\mathcal{N}_{i,j}$.

B. 2-pair Unicast Network Coding Problem

Definition 2.2: A 2-pair unicast problem is specified as follows.

- 1) A communication network $\mathcal{N} = (V, E, \{s_1, s_2\}, \{t_1, t_2\})$.
- 2) Two desired unit flows from s_i to t_i for $i = 1, 2$.

Note that the underlying network $\mathcal{N} = (V, E, \{s_1, s_2\}, \{t_1, t_2\})$ is usually called a *2-pair (unicast) network* in this paper. The desired flows, which are generated in s_i and to be recovered in t_i , for $i = 1, 2$, are considered as independent random variables with unit entropies and denoted by X_1 and X_2 , respectively. The information transformation is assumed to be delay-free and error-free. The information transmitted over an edge e and an edge set A are considered as random variables and denoted by X_e and X_A , respectively. The entropies of X_e and X_A are simply denoted by $H(e)$ and $H(A)$, respectively.

Without loss of generality, we add an auxiliary source node with a single out-edge (denoted by $S(i)$ for $i = 1, 2$) to each source node and add an auxiliary sink node with a single in-edge (denoted by $T(i)$ for $i = 1, 2$) from each sink node. For convenience, the edges of $S(i)$ and $T(i)$ are called *the information edges*, since they are responsible for delivering and/or recovering the original information. Thus in this paper, each source node s_i is assumed to have one out-edge and no in-edge, and each sink node t_i is assumed to have one in-edge and no out-edge. We also assume that each node except s_i and t_i , for $i = 1, 2$, has at least one in-edge and one out-edge.

The information edges $S(i)$ and $T(i)$ can be assumed to have capacity $C(s_i, t_i)$ in order to maintain the maximum flows from s_i to t_i for $i = 1, 2$. But in Section IV-B, information edges are assumed to have unit capacity since the desired information flows have unit rates. Except for the information edges, all the other edges are assumed to have unit capacity.

A *network code* assigned to a 2-pair unicast network $\mathcal{N} = (V, E, \{s_1, s_2\}, \{t_1, t_2\})$ is defined as a collection of functions $\{f_e : e \in E\}$ such that $X_e = f_e(X_{\text{In}(e)})$, where $\text{In}(e) = \{e' \in E : \text{head}(e') = \text{tail}(e)\}$ (when $e = S(i)$, then $\text{In}(e) = \emptyset$, and let $X_{S(i)} = X_i$ for $i = 1, 2$). A *network coding solution* for a 2-pair unicast network is a network code such that $H(S(i)|T(i)) = 0$ for $i = 1, 2$. A 2-pair unicast problem is called *solvable* when a network coding solution exists (the underlying 2-pair unicast network is called *available*), and called *unsolvable* (the underlying 2-pair unicast network is called *unavailable*) otherwise.

Remark 2.3: By the definition, for any network code $\{f_e : e \in E\}$, the condition that X_e is a function of $X_{\text{In}(e)}$ indicates that $H(X_e|X_{\text{In}(e)}) = 0$.

Unlike the definition of a network coding solution in [8], where an arbitrary positive network coding rate is considered, the 2-pair unicast problem here aims at supporting two unit rate flows. Hence, the definition of a network coding solution has been slightly changed. In fact, it corresponds to the network coding solution in [8] with rate ≥ 1 .

C. Path Combination/Decomposition

A (simple) path can be represented as a string of ordered edges, $P = (e_1, e_2, \dots, e_n)$, with $\text{head}(e_i) = \text{tail}(e_{i+1})$ for $i = 1, 2, \dots, n-1$, where e_i is called an *up-link* (*down-link*) of e_j if $i < j$ ($i > j$). We use $e \in P$ to denote an edge e lies in a path P . For a DAG, it is widely known that there exists a *topological order* for the edges according to the up- (or down-) link relation, that is, if e_i is an up-link of e_j for some path P , then e_i is an up-link of e_j for any path Q for $e_i, e_j \in Q$. This topological order of the edges of a DAG will always be used in this paper.

A frequently used technique in this paper is path combination/decomposition. We denote $P[v_i, v_j]$ as the section of P from node v_i to node v_j . Similarly, $P[e_i, e_j]$ is used to denote the section of P from $\text{tail}(e_i)$ to $\text{head}(e_j)$, where e_i and e_j are two different edges in P . We also use $P[e_i, v_j]$ and $P[v_i, e_j]$ to denote the sections of P from $\text{tail}(e_i)$ to node v_j , and from node v_i to $\text{head}(e_j)$, respectively. Let $P_1 = (e_1, e_2, \dots, e_n)$ and $P_2 = (e'_1, e'_2, \dots, e'_m)$ be two paths such that $\text{head}(P_1) = \text{tail}(P_2)$ (that is, $\text{head}(e_n) = \text{tail}(e'_1)$). We denote the path $P = (e_1, e_2, \dots, e_n, e'_1, \dots, e'_m)$ as P_1 - P_2 . Similarly, we use P - $P^{(k)}$ to denote the configuration by joining a simple path P and an edge-disjoint k -path $P^{(k)}$. An edge-disjoint k -path composed by s - t paths P_1, P_2, \dots, P_k is sometimes denoted as $P^{(k)} = P_1 \cup P_2 \cup \dots \cup P_k$. Moreover, a path is usually regarded as a collection of edges. For example, we use $P \cup Q$ and $P \cap Q$ to represent the union and the intersection (of the edges) of paths P and Q , respectively.

III. NETWORK STRUCTURE ANALYSIS

In this section, we explore the underlying structure of 2-pair unicast networks. In the following, the 2-pair network will be assumed with $C(s_1, t_1) = C(s_2, t_2) = 1$. For the case $C(s_1, t_1) \cdot C(s_2, t_2) \geq 2$, it will be discussed later (If $C(s_1, t_1) \cdot C(s_2, t_2) = 0$, then there is no path from s_1 to t_1 or from s_2 to t_2 , and the 2-pair unicast problem is unsolvable obviously.).

Throughout the paper, the terms “ \mathcal{N} has underlying network \mathcal{N}_0 ,” “ \mathcal{N} contains a copy of \mathcal{N}_0 ,” or simply “ \mathcal{N} contains \mathcal{N}_0 ” will be equivalently used to indicate the existence of a same topology between paths of \mathcal{N} and edges of \mathcal{N}_0 . Formally, we give the following definition.

Definition 3.1: Let $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$ and $\mathcal{N}_0 = (V', E', \{s'_1, t'_1\}, \{s'_2, t'_2\})$ be two 2-pair unicast networks. We say \mathcal{N} contains a copy of \mathcal{N}_0 if there exists a function f from the edges of \mathcal{N}_0 to the paths of \mathcal{N} satisfying:

- (1) If $\text{tail}(e') = s'_i$, then $\text{tail}(f(e')) = s_i$, for $e' \in E'$ and $i = 1, 2$;
- (2) If $\text{head}(e') = t'_i$, then $\text{head}(f(e')) = t_i$, for $e' \in E'$ and $i = 1, 2$;
- (3) If $\text{head}(e'_1) = \text{tail}(e'_2)$, then $\text{head}(f(e'_1)) = \text{tail}(f(e'_2))$, for $e'_1, e'_2 \in E'$;
- (4) If $e'_1 \neq e'_2$, then $f(e'_1)$ and $f(e'_2)$ are edge-disjoint, for $e'_1, e'_2 \in E'$.

Obviously, this definition can be generalized to an arbitrary multi-source multi-sink network, as similar to the notion of *subgraph homeomorphism* in graph theory (see [14]). Generally, paths under the subgraph homeomorphism are needed to be node-disjoint, which is naturally loosened here to edge-disjoint since the network information flow problem concentrates on the link capacity constraints. Before illustrating the main results, we give a lemma.

Lemma 3.2: Let $\mathcal{N} = (V, E, s, t)$ be a point-to-point network such that $C(s, t) = 1$. Denote \mathcal{A} as its \mathcal{A} -set. Assume that s has a unique out-edge, $S(1)$, and t has a unique in-edge, $T(1)$. Then the following items hold.

- 1) For any edge $e \in \mathcal{A}$ and any s - t path P , we have $e \in P$;
- 2) For edge $e \notin \mathcal{A}$, there exists an s - t path P such that $e \notin P$;
- 3) \mathcal{N} has a subnetwork $\mathcal{N}_0 = P_1 - P_1^{(2)} - P_2 - P_2^{(2)} - \dots - P_n^{(2)} - P_{n+1}$ such that $\mathcal{A} = P_1 \cup P_2 \cup \dots \cup P_{n+1}$, where $\text{tail}(P_1) = s$, $\text{head}(P_{n+1}) = t$, and path P_i is regarded as the collection of edges.

Proof: The first two items are obvious by Proposition 2.1. Now we prove 3) by constructing \mathcal{N}_0 . Let $\mathcal{A} = \{e_1, e_2, \dots, e_m\}$ such that e_i is an up-link of e_j for $i < j$ (where, $e_1 = S(1)$, and $e_m = T(1)$). Let $e_i, e_{i+1} \in \mathcal{A}$ and $\text{head}(e_i) \neq \text{tail}(e_{i+1})$. Note that $(V, E, \text{head}(e_i), \text{tail}(e_{i+1}))$ is also a point-to-point network.

Consider $C(\text{head}(e_i), \text{tail}(e_{i+1}))$. If $C(\text{head}(e_i), \text{tail}(e_{i+1})) = 1$, then there exists a $\text{head}(e_i)$ - $\text{tail}(e_{i+1})$ minimum cut which contains only one edge, namely, $\{e\}$. Since e is a down-link of e_i and an up-link of e_{i+1} , we have $e \notin \mathcal{A}$. On the other hand, by Proposition 2.1, any $\text{head}(e_i)$ - $\text{tail}(e_{i+1})$ path must pass through e . Thus any s - t path must pass through e . By Proposition 2.1, $e \in \mathcal{A}$, which contradicts to $e \notin \mathcal{A}$. Therefore $C(\text{head}(e_i), \text{tail}(e_{i+1})) \geq 2$.

Take an edge-disjoint 2-path from $\text{head}(e_i)$ to $\text{tail}(e_{i+1})$, and denote it as $Q_i^{(2)}$. Suppose that $e_{i_1}, e_{i_2}, \dots, e_{i_n}$ are all the links of \mathcal{A} with $\text{head}(e_{i_k}) \neq \text{tail}(e_{i_{k+1}})$. Let $P_1 = (e_1, \dots, e_{i_1})$, $P_k = (e_{i_{k-1}+1}, \dots, e_{i_k})$, for $k = 2, 3, \dots, n$, $P_{n+1} = (e_{i_n+1}, \dots, e_m)$, and $P_k^{(2)} = Q_{i_k}^{(2)}$ for $k = 1, 2, \dots, n$. Then $\mathcal{N}_0 = P_1 - P_1^{(2)} - P_2 - P_2^{(2)} - \dots - P_n^{(2)} - P_{n+1}$ satisfies the desired conditions. The proof is done. \blacksquare

By observing the proof process of Lemma 3.2, we get the following corollary.

Corollary 3.3: Let $\mathcal{N} = (V, E, s, t)$ be a point-to-point network such that $C(s, t) = 1$ with \mathcal{A} -set $\mathcal{A} = \{e_1, e_2, \dots, e_n\}$. If $\text{head}(e_i) \neq \text{tail}(e_{i+1})$ for some $1 \leq i < n$, then there exists an edge-disjoint 2-path from $\text{head}(e_i)$ to $\text{tail}(e_{i+1})$.

Now we start to discuss the characteristics of a 2-pair unicast network with $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \emptyset$. Note that $\mathcal{A}_{i,j}$ is the \mathcal{A} -set of the point-to-point network $\mathcal{N}_{i,j} = (V, E, s_i, t_j)$, for $i, j = 1, 2$.

Theorem 3.4: Let $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$ be a 2-pair unicast network with $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \emptyset$. Then there is either an s_1 - t_1 path disjoint with $\mathcal{A}_{2,2}$ or an s_2 - t_2 path disjoint with $\mathcal{A}_{1,1}$.

Proof: Let $\mathcal{A}_{1,1} = \{e_1, e_2, \dots, e_n\}$ such that e_i is an up-link of e_j for $i < j$ and let P_1 be an s_1 - t_1 path. If $P_1 \cap \mathcal{A}_{2,2} = \emptyset$, then we are done. Now suppose P_1 contains an edge $e^* \in \mathcal{A}_{2,2}$. Fix m , $0 \leq m \leq n$, such that e^* is a down-link of e_i for $i \leq m$ and an up-link of e_i for $i > m$. We can construct an s_2 - t_2 path disjoint with $\mathcal{A}_{1,1}$ as follows.

If $m > 0$, then we can find an s_2 - t_2 path P_2 not containing e_m since $e_m \notin \mathcal{A}_{2,2}$. (if $m = 0$, then P_2 can be any s_2 - t_2 path.) Since $e^* \in \mathcal{A}_{2,2}$, e^* lies on P_2 . The path $P_2[s_2, e^*]$ cannot contain edges e_i for $i < m$,

because, if it did, then $P_1[s_1, e_i] - P_2[\text{head}(e_i), e^*] - P_1[\text{head}(e^*), t_1]$ would be an s_1 - t_1 path not containing e_m , which contradicts to $e_m \in \mathcal{A}_{1,1}$. Also $P_2[s_2, e^*]$ cannot contain any edge e_j with $j > m$ because this would make e_j an up-link of e^* in P_2 and a down-link of e^* in P_1 . Thus $P_2[s_2, e^*] \cap \mathcal{A}_{1,1} = \emptyset$.

Similarly, if $m < n$, we can find an s_2 - t_2 path P'_2 not containing e_{m+1} . (If $m = n$, P'_2 can be any s_2 - t_2 path.) A similar argument as above shows that $P'_2[e^*, t_2] \cap \mathcal{A}_{1,1} = \emptyset$.

Combining $P_2[s_2, e^*]$ and $P'_2[e^*, t_2]$ together, we have an s_2 - t_2 path $P_2[s_2, e^*] - P'_2[\text{head}(e^*), t_2]$, which is disjoint with $\mathcal{A}_{1,1}$. The proof is completed. \blacksquare

Theorem 3.5: Let $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$ be a 2-pair unicast network. If $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \emptyset$, then the network contains Fig.1(a), Fig.1(b), or Fig.1(c).

Proof: By Theorem 3.4, we first assume that there exists an s_2 - t_2 path disjoint with $\mathcal{A}_{1,1}$, and prove that the network contains Fig.1(a) or Fig.1(b).

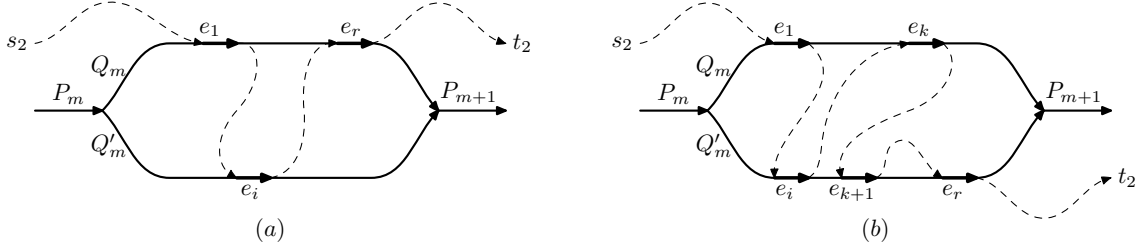
By Lemma 3.2, let $\mathcal{N}_0 = P_1 - P_1^{(2)} - P_2 - P_2^{(2)} - \dots - P_{n-1}^{(2)} - P_n$ be a subnetwork of \mathcal{N} such that $\mathcal{A}_{1,1} = P_1 \cup P_2 \cup \dots \cup P_n$ with $P_i^{(2)} = Q_i \cup Q'_i$ for $i = 1, 2, \dots, n-1$. Let P be an s_2 - t_2 path disjoint with $\mathcal{A}_{1,1}$. If $P \cap \mathcal{N}_0 = \emptyset$, then \mathcal{N} contains Fig.1(a) since $P_1 - Q_1 - P_2 - Q_2 - \dots - Q_{n-1} - P_n$ and P are edge-disjoint s_1 - t_1 and s_2 - t_2 paths. If $P \cap \mathcal{N}_0 \neq \emptyset$, then assume $e^* \in P \cap \mathcal{N}_0$ and let $e^* \in Q_m$ for some $1 \leq m \leq n-1$. We now prove that \mathcal{N} contains Fig.1(a) or Fig.1(b).

We claim first that $P \cap P_i^{(2)} = \emptyset$ for $i \neq m$. If it is not true, without loss of generality, assume $e' \in P \cap Q_i$ and consider the following two cases. (1) $i < m$. Since e' is an up-link of e^* in $P_1 - Q_1 - P_2 - Q_2 - \dots - Q_{n-1} - P_n$, e' is an up-link of e^* according to P . So $P_1 - Q_1 - \dots - Q_i[\text{tail}(Q_i), e'] - P[e', e^*] - Q_m[e^*, \text{head}(Q_m)] - P_{m+1} - \dots - P_n$ is an s_1 - t_1 path disjoint with P_m , which contradicts to $P_m \subset \mathcal{A}_{1,1}$. (2) $i > m$. Similarly, one can see that s_1 - t_1 path $P_1 - Q_1 - \dots - Q_m[\text{tail}(Q_m), e^*] - P[e^*, e'] - Q_i[e', \text{head}(Q_i)] - P_{i+1} - \dots - P_n$ is disjoint with $P_i \subset \mathcal{A}_{1,1}$, a contradiction.

Now assume that $\mathcal{N}_0 \cap P = P_m^{(2)} \cap P = (Q_m \cup Q'_m) \cap P = \{e_1, e_2, \dots, e_r\}$ such that e_j is a down-link of e_i for $1 \leq i < j \leq r$. Consider the following cases:

- 1) If $e_1, e_r \in Q_m$, as shown in Fig.2(a), then s_2 - t_2 path $P[s_2, \text{tail}(e_1)] - Q_m[e_1, e_r] - P[\text{head}(e_r), t_2]$ is edge-disjoint with s_1 - t_1 path $P_1 - Q_1 - \dots - P_m - Q'_m - P_{m+1} - \dots - Q_{n-1} - P_n$. The network contains Fig.1(a).
- 2) If $e_1 \in Q_m$ and $e_r \in Q'_m$, then let k be an index such that $e_k \in Q_m$, and $e_{k'} \in Q'_m$ for all $k' > k$, as shown in Fig.2(b). It can be checked that the network contains Fig.1(b) with the function f :
 $(s_1, v_1) \mapsto P_1 - Q_1 - \dots - P_m$; $(s_2, v_2) \mapsto P[s_2, \text{tail}(e_1)]$; $(v_6, t_1) \mapsto P_{m+1} - Q_{m+1} - \dots - P_n$; $(v_5, t_2) \mapsto P[\text{head}(e_r), t_2]$; $(v_1, v_2) \mapsto Q_m[\text{tail}(Q_m), \text{tail}(e_1)]$; $(v_1, v_4) \mapsto Q'_m[\text{tail}(Q'_m), \text{tail}(e_{k+1})]$; $(v_2, v_3) \mapsto Q_m[\text{tail}(e_1), \text{head}(e_k)]$; $(v_3, v_4) \mapsto P[\text{head}(e_k), \text{tail}(e_{k+1})]$; $(v_4, v_5) \mapsto Q'_m[\text{tail}(e_{k+1}), \text{head}(e_r)]$; $(v_5, v_6) \mapsto Q'_m[\text{head}(e_r), \text{head}(Q'_m)]$. The imaged paths are edge-disjoint because any two disjoint sections of P (and \mathcal{N}_0) are edge-disjoint, and $P \cap \mathcal{N}_0 = P \cap P_m^{(2)}$.
- 3) If $e_1, e_r \in Q'_m$, the discussion is similarly to that of case 1). The network contains Fig.1(a).
- 4) If $e_1 \in Q'_m$ and $e_r \in Q_m$, the discussion is similar to that of case 2). The network contains Fig.1(b).

In the case where there exists an s_1 - t_1 path disjoint with $\mathcal{A}_{2,2}$, one can prove symmetrically that the network contains Fig.1(a) or Fig.1(c). \blacksquare

Fig. 2: The relationship between P and \mathcal{N}_0 .

(a): The case of $e_1, e_r \in Q_m$. (b): The case of $e_1 \in Q_m, e_r \in Q'_m$.

In the above discussions, we have deduced the underlying structure of the 2-pair unicast network with $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \emptyset$. Now we deal with the 2-pair networks with $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$. Firstly, we need a lemma.

Lemma 3.6: Let $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$ be a 2-pair unicast network such that $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$, and let $\mathcal{A}_{1,1} = \{e_1, e_2, \dots, e_n\}$. If $e_i, e_j \in \mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}$ ($i < j$), then $e_\ell \in \mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}$ for $i < \ell < j$.

Proof: Assume $e_\ell \notin \mathcal{A}_{2,2}$ and let Q be an s_2 - t_2 path not containing e_ℓ . Then, for any s_1 - t_1 path P , we have an s_1 - t_1 path $P' = P[s_1, \text{tail}(e_i)] - Q[e_i, e_j] - P[\text{head}(e_j), t_1]$ not containing e_ℓ . Therefore $e_\ell \notin \mathcal{A}_{1,1}$, a contradiction. \blacksquare

Given a 2-pair unicast network $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$, an s_1 - t_1 path P and an s_2 - t_2 path Q , by Lemma 3.2, one can have that $P \supseteq \mathcal{A}_{1,1}$ and $Q \supseteq \mathcal{A}_{2,2}$, and thus $P \cap Q \supseteq \mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}$. Moreover, when $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$, one can prove further that there exist an s_1 - t_1 path P and an s_2 - t_2 path Q such that $P \cap Q = \mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}$ as the following theorem shows.

Theorem 3.7: Let $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$ be a 2-pair unicast network such that $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$. Then there exist an s_1 - t_1 path P and an s_2 - t_2 path Q such that $P \cap Q = \mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}$.

Proof: We construct P and Q by using the technique of path combination (see Fig.3(a)). By Lemma 3.6, one can let $\mathcal{A}_{1,1} = \{e_1, e_2, \dots, e_n\}$ and $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \{e_m, e_{m+1}, \dots, e_{m+j}\}$. Moreover, we have that $m \geq 2$ and $m + j \leq n - 1$ by the assumptions that s_i has a single out-edge and t_i has a single in-edge. Denote $\text{tail}(e_m) = s$ and $\text{head}(e_{m+j}) = t$. We claim that there exist an s_1 - s path \hat{P} and an s_2 - s path \hat{Q} such that $\hat{P} \cap \hat{Q} = \emptyset$.

To prove this, let \hat{P}' be an arbitrary s_1 - t_1 path. By Lemma 3.2, one can take an s_2 - t_2 path \hat{Q}' such that $e_{m-1} \notin \hat{Q}'$. If there is an $e^* \in \hat{P}'[s_1, e_{m-1}] \cap \hat{Q}'[s_2, s]$, then $\hat{P}'[s_1, e^*] - \hat{Q}'[\text{head}(e^*), s] - \hat{P}'[s, t_1]$ is an s_1 - t_1 path not containing e_{m-1} . So $e_{m-1} \notin \mathcal{A}_{1,1}$, resulting in a contradiction. Thus there exist an s_1 - $\text{head}(e_{m-1})$ path $\hat{P}'[s_1, e_{m-1}]$ and an s_2 - s path $\hat{Q}'[s_2, s]$ with $\hat{P}'[s_1, e_{m-1}] \cap \hat{Q}'[s_2, s] = \emptyset$. If $\text{head}(e_{m-1}) = s$, then we are done by letting $\hat{P} = \hat{P}'[s_1, s]$ and $\hat{Q} = \hat{Q}'[s_2, s]$. Now suppose that $\text{head}(e_{m-1}) \neq s$. By Lemma 3.2, there exists an edge-disjoint 2-path $P^{(2)}$ from $\text{head}(e_{m-1})$ to s . Let $P^{(2)} = Q_1 \cup Q_2$. If $\hat{Q}' \cap P^{(2)} = \emptyset$, then we are done by letting $\hat{P} = \hat{P}'[s_1, e_{m-1}] - Q_1$ and $\hat{Q} = \hat{Q}'[s_2, s]$. If $\hat{Q}' \cap P^{(2)} \neq \emptyset$, let $\{\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_\ell\} \in \hat{Q}' \cap P^{(2)}$ such that \tilde{e}_j is a down-link of \tilde{e}_i for $i < j$. Without loss of generality, assume that $\tilde{e}_1 \in Q_1$. Then $\hat{P} = \hat{P}'[s_1, e_{m-1}] - Q_2$ is an s_1 - s path and $\hat{Q} = \hat{Q}'[s_2, \tilde{e}_1] - Q_1[\text{head}(\tilde{e}_1), s]$ is an s_2 - s path satisfy $\hat{P} \cap \hat{Q} = \emptyset$.

Similarly, one can find a t - t_1 path \check{P} and a t - t_2 path \check{Q} with $\check{P} \cap \check{Q} = \emptyset$.

Let $e_{i_1}, e_{i_2}, \dots, e_{i_n}$ be all the links such that $\text{head}(e_{i_k}) \neq \text{tail}(e_{i_{k+1}})$ for $m \leq i_k < m + j$. Noticing that $e_{i_k}, e_{i_{k+1}} \in \mathcal{A}_{1,1}$, there exist an edge-disjoint 2-path, namely, $\bar{P}_k^{(2)} = \bar{Q}_k \cup \bar{Q}'_k$ from $\text{head}(e_{i_k})$ to

$\text{tail}(e_{i_k+1})$ by Corollary 3.3. Let $\bar{P}_1 = (e_m, \dots, e_{i_1})$, $\bar{P}_k = (e_{i_{k-1}+1}, \dots, e_{i_k})$ for $k = 2, 3, \dots, n$, and $\bar{P}_{n+1} = (e_{i_n+1}, \dots, e_{m+j})$. Set $\bar{P} = \bar{P}_1 - \bar{Q}_1 - \bar{P}_2 - \bar{Q}_2 - \dots - \bar{Q}_n - \bar{P}_{n+1}$ and $\bar{Q} = \bar{P}_1 - \bar{Q}'_1 - \bar{P}_2 - \bar{Q}'_2 - \dots - \bar{Q}'_n - \bar{P}_{n+1}$. We have $\bar{P} \cap \bar{Q} = \{e_m, e_{m+1}, \dots, e_{m+j}\} = \mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}$.

Let $P = \hat{P} - \bar{P} - \check{P}$ and $Q = \hat{Q} - \bar{Q} - \check{Q}$. Then P is an s_1 - t_1 path and Q is an s_2 - t_2 path such that $P \cap Q = \mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}$, which completes the proof. ■

Corollary 3.8: Let $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$ be a 2-pair unicast network such that $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$. Then \mathcal{N} contains a copy of the network as shown in Fig.3(b).

Proof: Using the notations in the proof of Theorem 3.7, a function f can be assigned from the edges of Fig.3(b) to the paths of Fig.3(a) such that $(s_1, v_1) \mapsto \hat{P}$; $(s_2, v_1) \mapsto \hat{Q}$; $(v_1, v_2) \mapsto \bar{P}$; $(v_2, t_1) \mapsto \check{P}$; and $(v_2, t_2) \mapsto \check{Q}$. The imaged paths are edge-disjoint because: (1) the edges in \hat{P} and in \hat{Q} are up-links of the edges in \bar{P} ; (2) the edges in \bar{P} are up-links of the edges in \check{P} and in \check{Q} ; and (3) $\hat{P} \cap \hat{Q} = \check{P} \cap \check{Q} = \emptyset$. ■

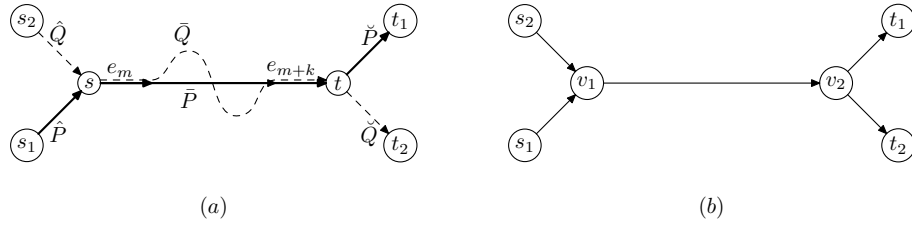


Fig. 3: The case $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$.

(a): The construction of paths P and Q such that $P \cap Q = \mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}$ with P in bold line and Q in dashed line.

(b): The underlying network for the 2-pair unicast network with $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$.

Based on Theorem 3.7, we have the following theorem.

Theorem 3.9: Let $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$ be a 2-pair unicast network such that $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$ and there exist an s_1 - t_2 path P_1 and an s_2 - t_1 path P_2 with $P_i \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$ for $i = 1, 2$. Then \mathcal{N} contains Fig.1(d).

Proof: Let $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \{e_1, e_2, \dots, e_k\}$. By Theorem 3.7, there exist an s_1 - t_1 path P and an s_2 - t_2 path Q such that $P \cap Q = \{e_1, e_2, \dots, e_k\}$. Let P_1 be an s_1 - t_2 path and P_2 be an s_2 - t_1 path such that $P_i \cap \mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \emptyset$ ($i = 1, 2$). We prove firstly the following properties of P , Q , P_1 and P_2 and then prove \mathcal{N} contains Fig.1(d).

- 1) $P_1 \cap P[e_1, t_1] = \emptyset$, $P_1 \cap Q[s_2, e_k] = \emptyset$;
- 2) $P_2 \cap P[s_1, e_k] = \emptyset$, $P_2 \cap Q[e_1, t_2] = \emptyset$;
- 3) $P_1 \cap P_2 = \emptyset$.

We prove them one by one.

- 1) Suppose that $P_1 \cap P[e_1, t_1] \neq \emptyset$. Let $e \in P_1 \cap P[e_1, t_1]$. Then $P_1[s_1, e] - P[\text{head}(e), t_1]$ is an s_1 - t_1 path not containing $e_1 \in \mathcal{A}_{1,1}$, resulting in a contradiction. Thus $P_1 \cap P[e_1, t_1] = \emptyset$. Similarly, if $e \in P_1 \cap Q[s_2, e_k]$ for some edge e , then $Q[s_2, e] - P_1[\text{head}(e), t_2]$ is an s_2 - t_2 path without passing through e_1 , which contradicts to $e_1 \in \mathcal{A}_{2,2}$.

- 2) It can be proved similarly.

- 3) Suppose that $P_1 \cap P_2 \neq \emptyset$, and let $e' \in P_1 \cap P_2$. Then, $P_1[s_1, e']$ - $P_2[head(e'), t_1]$ is an s_1 - t_1 path not containing e_1 , which is a contradiction to $e_1 \in \mathcal{A}_{1,1}$.

By property 1), we can assume that $P_1 \cap P = P_1 \cap P[s_1, tail(e_1)] = \{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n\}$ and $P_1 \cap Q = P_1 \cap Q[head(e_k), t_2] = \{\check{e}_1, \check{e}_2, \dots, \check{e}_m\}$, (both in the topological order). It can be seen that (1) $\{\hat{e}_1, \hat{e}_2, \dots, \hat{e}_n\} \neq \emptyset$ and $\{\check{e}_1, \check{e}_2, \dots, \check{e}_m\} \neq \emptyset$; and (2) $head(\hat{e}_n) \neq tail(e_1)$ and $head(e_k) \neq tail(\check{e}_1)$. In fact, (1) holds since $\hat{e}_1 = S(1)$ is the unique out-edge of s_1 and $\check{e}_m = T(2)$ is the unique in-edge of t_2 . For the property (2), if $head(\hat{e}_n) = tail(e_1)$, then $Q[s_2, tail(e_1)]$ - $P_1[head(\hat{e}_n), t_2]$ is an s_2 - t_2 path disjoint with $\mathcal{A}_{2,2}$, while if $head(e_k) = tail(\check{e}_1)$, then $P_1[s_1, tail(\check{e}_1)]$ - $P[head(e_k), t_1]$ is an s_1 - t_1 path disjoint with $\mathcal{A}_{1,1}$. Both are contradictions.

Similarly, one can prove that $P_2 \cap Q = P_2 \cap Q[s_2, tail(e_1)] \neq \emptyset$ and $P_2 \cap P = P_2 \cap P[head(e_k), t_1] \neq \emptyset$. Let $P_2 \cap Q = P_2 \cap Q[s_2, tail(e_1)] = \{\hat{e}'_1, \hat{e}'_2, \dots, \hat{e}'_u\}$ and let $P_2 \cap P = P_2 \cap P[head(e_k), t_1] = \{\check{e}'_1, \check{e}'_2, \dots, \check{e}'_v\}$. We have $head(\hat{e}'_u) \neq tail(e_1)$ and $head(e_k) \neq tail(\check{e}'_1)$.

Now we can define a function f from the edges of Fig.1(d) to the paths P, Q, P_1, Q_1 of \mathcal{N} (see Fig.4):
 $(s_1, v_1) \mapsto P[s_1, \hat{e}_n]$; $(s_2, v_2) \mapsto Q[s_2, \hat{e}'_u]$; $(v_1, v_3) \mapsto P[head(\hat{e}_n), tail(e_1)]$; $(v_2, v_3) \mapsto Q[head(\hat{e}'_u), tail(e_1)]$;
 $(v_3, v_4) \mapsto P[e_1, e_k]$; $(v_1, v_5) \mapsto P_1[head(\hat{e}_n), tail(\check{e}_1)]$; $(v_2, v_6) \mapsto P_2[head(\hat{e}'_u), tail(\check{e}'_1)]$; $(v_4, v_5) \mapsto Q[head(e_k), tail(\check{e}_1)]$; $(v_4, v_6) \mapsto P[head(e_k), tail(\check{e}'_1)]$; $(v_6, t_1) \mapsto P[tail(\check{e}'_1), t_1]$; $(v_5, t_2) \mapsto Q[tail(\check{e}_1), t_2]$.
 Obviously, f results in disjoint paths. The theorem is proved. \blacksquare

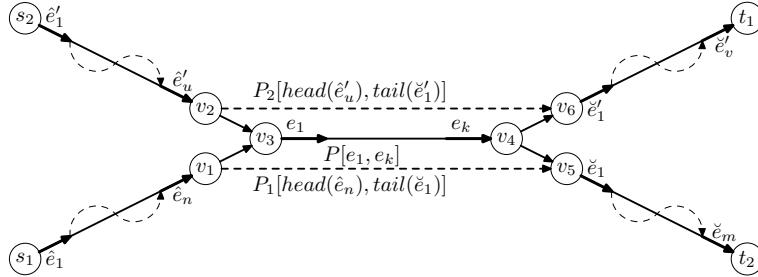


Fig. 4: The figure illustrating the proof of Theorem 3.9.

In the figure, the path sections of P and Q (P_1 and P_2) are shown in bold (dashed) lines.

We discussed the structures of 2-pair unicast networks with $C(s_1, t_1) \cdot C(s_2, t_2) = 1$ previously. For the network with $C(s_1, t_1) \cdot C(s_2, t_2) \geq 2$, its structure can be deduced directly from Theorem 3.5.

Corollary 3.10: Let $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$ be a 2-pair unicast network. If $C(s_1, t_1) \cdot C(s_2, t_2) \geq 2$, then \mathcal{N} contains a copy of the networks Fig.1(a), Fig.1(b), or Fig.1(c).

Proof: Without loss of generality, we assume that $C(s_1, t_1) \geq 2$. By the prior assumptions, s_i (t_i) has the unique out-edge $S(i)$ (in-edge $T(i)$) with capacity $C(s_i, t_i)$ for $i = 1, 2$, and except for these four edges, all the other edges have unit capacities. Then the Max-flow Min-cut theorem implies that there exist an edge-disjoint 2-path $P^{(2)}$ from $head(S(1))$ to $tail(T(1))$. Let $P^{(2)} = Q \cup Q'$ and take an s_2 - t_2 path P . If $P^{(2)} \cap P = \emptyset$, then \mathcal{N} contain Fig.1(a) by noticing that $S(1)$ - Q - $T(1)$ and P are edge-disjoint. Now assume $P^{(2)} \cap P = \{e_1, e_2, \dots, e_r\}$. Similar to the latter part of the proof of Theorem 3.5, there are 4 cases need to be discussed (A figure to illustrate these cases is a minor modification on Fig.2 by replacing Q_m , Q'_m , P_m and P_{m+1} with Q , Q' , $S(1)$, and $T(1)$ respectively):

- 1) If $e_1, e_r \in Q$, then $S(1)$ - Q' - $T(1)$ is an s_1 - t_1 path which is edge-disjoint with the s_2 - t_2 path $P[s_2, \text{tail}(e_1)]$ - $Q[e_1, e_r]$ - $P[\text{head}(e_r), t_2]$. The network contains Fig.1(a).
 - 2) If $e_1 \in Q$ and $e_r \in Q'$, let k be the maximum index such that $e_k \in Q$ and $e_{k+1} \in Q'$ and let f be defined as $(s_1, v_1) \mapsto S(1)$; $(s_2, v_2) \mapsto P[s_2, \text{tail}(e_1)]$; $(v_6, t_1) \mapsto T(1)$; $(v_5, t_2) \mapsto P[\text{head}(e_r), t_2]$; $(v_1, v_2) \mapsto Q[\text{tail}(Q), \text{tail}(e_1)]$; $(v_1, v_4) \mapsto Q'[\text{tail}(Q'), \text{tail}(e_{k+1})]$; $(v_2, v_3) \mapsto Q[e_1, e_k]$; $(v_3, v_4) \mapsto P[\text{head}(e_k), \text{tail}(e_{k+1})]$; $(v_4, v_5) \mapsto Q'[e_{k+1}, e_r]$; $(v_5, v_6) \mapsto Q'[\text{head}(e_r), \text{head}(Q')]$. The network contains Fig.1(b).
 - 3) If $e_1, e_r \in Q'$, then the network contains Fig.1(a), which is similar to case 1).
 - 4) If $e_1 \in Q'$ and $e_r \in Q$, then the network contains Fig.1(b), which is similar to case 2).
- Likewise, if $C(s_2, t_2) \geq 2$, similar discussions can conclude that the network contains Fig.1(a) or Fig.1(c). ■

IV. SOLVABILITY ANALYSIS

In this section, we apply those structural results in Section III to analyze the capacity of 2-pair unicast networks. Those results deduce a complete classification of the 2-pair unicast available networks (Theorem 4.3), and an efficient algorithm to determine the solvability of a 2-pair unicast problems (Algorithm 4.5). It meanwhile provides a new proof that linear network coding is sufficient for solving the 2-pair unicast problem (Corollary 4.6). Most importantly, It is showed that the solvability of a 2-pair unicast problem is completely decided by four subsets, $\mathcal{A}_{i,j}$ for $i, j = 1, 2$ of the underlying network (Theorem 4.8).

A. Solvability of 2-pair Unicast Problem

The results of this part are based on the technique of *informational dominance* in [8].

Definition 4.1 ([8]): Let $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$ be a 2-pair unicast network. We say an edge set A informationally dominates an edge set B if X_B is a function of X_A (or equivalently, $H(B|A) = 0$) for all network coding solutions, and denoted by $A \rightsquigarrow^i B$.

The informational dominance has the following properties [8]:

- 1) $T(i) \rightsquigarrow^i S(i)$, for $i = 1, 2$.
- 2) $A \rightsquigarrow^i A$, for $A \subseteq E$.
- 3) If $A \rightsquigarrow^i B$, and $A \rightsquigarrow^i C$, then $A \rightsquigarrow^i B \cup C$.
- 4) If $A \rightsquigarrow^i B$, and $B \rightsquigarrow^i C$, then $A \rightsquigarrow^i C$.
- 5) If B is downstream of A , then $A \rightsquigarrow^i B$, where B is *downstream* of A if there is no path from $S = \{s_1, s_2\}$ to B in $\mathcal{N} \setminus A$.

In the above, 1) holds by the definition of network coding solution; 2)-4) hold by the definition of informational dominance; As to 5), edge set B is called *downstream* of edge set A if there is no path from $S = \{s_1, s_2\}$ to B in $\mathcal{N} \setminus A$, the deduced network formed by \mathcal{N} deleting A (see [8]), and this item holds by observing that $X_e = f_e(X_{In(e)})$ for all $e \in E$ and all the paths from $S = \{s_1, s_2\}$ to B intersect A (a detailed proof can be found in Lemma 11, p.2353 of [8]).

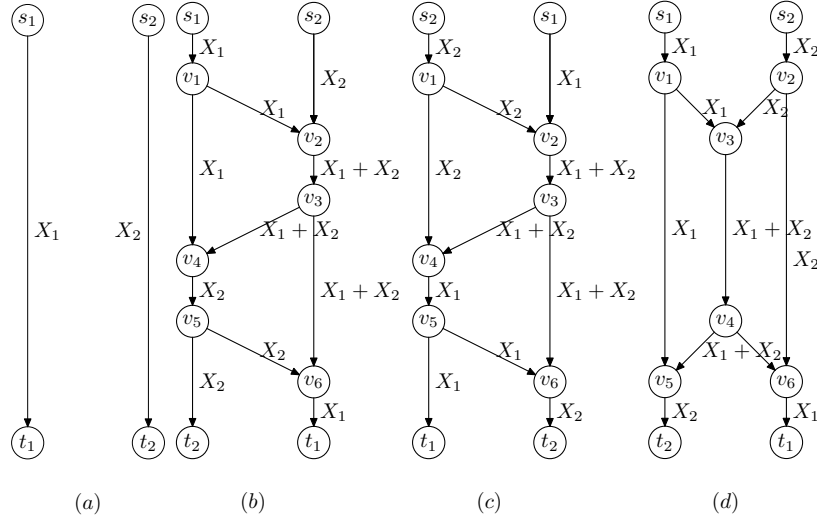


Fig. 5: Network coding solutions for Fig.1.

Given an arbitrary 2-pair unicast network $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$. If $C(s_1, t_1) \cdot C(s_2, t_2) \geq 2$, it contains a copy of Fig.1(a), Fig.1(b) or Fig.1(c). Thus \mathcal{N} is available by extending the network solution of Fig.5(a), Fig.5(b), or Fig.5(c) to the whole network. That is, to transmit X_e over the path $f(e)$ of \mathcal{N} , and not to transmit any signal over the other edges. When $C(s_1, t_1) \cdot C(s_2, t_2) = 1$ and $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \emptyset$, the network contains Fig.1(a), Fig.1(b), or Fig.1(c), and then it is available. When $C(s_1, t_1) \cdot C(s_2, t_2) = 1$ and $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$, we have,

Theorem 4.2: Let $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$ be a 2-pair unicast network such that $C(s_1, t_1) \cdot C(s_2, t_2) = 1$ and $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$. Then \mathcal{N} is available if and only if there exist an s_1 - t_2 path P_1 and an s_2 - t_1 path P_2 with $(P_1 \cup P_2) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$.

Proof: Let \mathcal{N} contain an s_1 - t_2 path P_1 and an s_2 - t_1 path P_2 with $(P_1 \cup P_2) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$. By Theorem 3.9, \mathcal{N} contains Fig.1(d). Then a network coding solution (shown in Fig.5(d)) can be extended to \mathcal{N} (by the aforementioned manner), and the sufficiency holds.

Suppose \mathcal{N} is available, and by Theorem 3.7, we take an s_1 - t_1 path P and an s_2 - t_2 path Q such that $P \cap Q = \mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}$. Without loss of generality, assume that no s_1 - t_2 path is disjoint with $\mathcal{A} = \mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}$, we prove the result by deduce a contradiction.

Let $\mathcal{A} = \{e_1, e_2, \dots, e_n\}$ (with the topological order) and take $e_i \in \mathcal{A}$, we claim that $T(2)$ is downstream of $\{e_i\}$. Firstly, there is no path from s_2 to t_2 in $\mathcal{N} \setminus \{e_i\}$ since $e_i \in \mathcal{A}_{2,2}$. Secondly, suppose that there exists an s_1 - t_2 path P_1 in $\mathcal{N} \setminus \{e_i\}$, then P_1 intersects \mathcal{A} . Let $e_j \in P_1 \cap \mathcal{A}$. If $i < j$, then $P_1[s_1, e_j]$ - $P_1[\text{head}(e_j), t_1]$ is an s_1 - t_1 path without passing through $e_i \in \mathcal{A}_{1,1}$, which is a contradiction. If $i > j$, then $Q[s_2, e_j]$ - $P_1[\text{head}(e_j), t_2]$ is an s_2 - t_2 path without passing through $e_i \in \mathcal{A}_{2,2}$, which is again a contradiction. Hence, there is neither s_2 - t_2 path nor s_1 - t_2 path in $\mathcal{N} \setminus \{e_i\}$, which implies that $T(2)$ is downstream of $\{e_i\}$. Moreover, one can have that $T(1)$ is downstream of $\{e_i\} \cup S(2)$ since all s_1 - t_1 paths intersect e_i and all s_2 - t_1 paths intersect $S(2)$.

Now we have already shown that $\{e_i\} \rightsquigarrow^i T(2)$, and $\{e_i\} \cup S(2) \rightsquigarrow^i T(1)$. Moreover, since $T(2) \rightsquigarrow^i S(2)$, one can have $\{e_i\} \rightsquigarrow^i S(2)$ by property 4). Thus $\{e_i\} \rightsquigarrow^i \{e_i\} \cup S(2) \rightsquigarrow^i T(1)$ by properties 2)-4). Using 3)

again, we have $\{e_i\} \rightsquigarrow^i T(1) \cup T(2)$, which contradicts to that e_i has unit capacity. The contradiction yields the necessity of the theorem. \blacksquare

The above discussions can conclude the following theorem.

Theorem 4.3: The 2-pair unicast problem is solvable if and only if the underlying network contains Fig.1(a), Fig.1(b), Fig.1(c) or Fig.1(d).

Remark 4.4: This theorem has been independently obtained by Chih-Chun Wang and Ness B. Shroff (Theorem 3 of [19]) by using different techniques. In [19], these underlying configurations were derived based on the *path overlap conditions* (Theorem 1 of [18]), which says that a 2-pair unicast problem is solvable if and only if it satisfies some path overlap conditions. Unlike [18], [19], we formulate the network structures by *cut set (A-set) relations*. The technical differences led to different algorithms for deciding the solvability of a 2-pair unicast problem, as follows.

Algorithm 4.5: (Checking the solvability of a 2-pair unicast problem.)

Input: A 2-pair unicast network $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$.

Output: The solvability of the 2-pair unicast problem.

(1) : Find $C(s_1, t_1)$ and $C(s_2, t_2)$, then calculate $C(s_1, t_1) \cdot C(s_2, t_2)$.

If $C(s_1, t_1) \cdot C(s_2, t_2) = 0$, \mathcal{N} is unavailable.

If $C(s_1, t_1) \cdot C(s_2, t_2) > 1$, \mathcal{N} is available.

If $C(s_1, t_1) \cdot C(s_2, t_2) = 1$, goto (2).

(2) : Find $\mathcal{A}_{1,1}$ and $\mathcal{A}_{2,2}$, then calculate $\mathcal{A} = \mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}$.

If $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \emptyset$, \mathcal{N} is available.

If $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$, goto (3).

(3) : Check the connectivity of s_1 to t_2 and s_2 to t_1 in $\mathcal{N}' = \mathcal{N} \setminus \mathcal{A}$.

If $C_{\mathcal{N}'}(s_1, t_2) \cdot C_{\mathcal{N}'}(s_2, t_1) = 0$, \mathcal{N} is unavailable.

If $C_{\mathcal{N}'}(s_1, t_2) \cdot C_{\mathcal{N}'}(s_2, t_1) \neq 0$, \mathcal{N} is available.

End.

In Algorithm 4.5, steps (1) and (2) can be finished in time $O(|V||E|^2)$ ([21]), and $O(|V||E|^3)$ ([20]), respectively. Step (3) can be done by a conventional breadth (or depth) first search algorithm with time $O(|V|^2)$. Note that the algorithm proposed in [18] and [19] (Corollary 1 of [18] and Corollary 1 of [19]) are based on the approach of [14] for finding k edge-disjoint paths. According to [14], one need to first calculate the *levels* of all the nodes, and then use a pebbling game for the path finding process. Comparing with this approach, Algorithm 4.5 is easier to implement.

Theorem 4.3 yields the following result, which was also independently pointed out in Corollary 2 of [18] and Corollary 3 of [19].

Corollary 4.6: Linear network coding is sufficient to solve the 2-pair unicast problem.

B. The 2-pair Unicast Networks with $C(s_i, t_j) = 1$

In this part, we consider the 2-pair unicast networks with $C(s_i, t_j) = 1$, for $i, j = 1, 2$.

Lemma 4.7: Let $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$ be a 2-pair unicast network with $C(s_i, t_j) = 1$ for $i, j = 1, 2$, and $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$. Then there exist an s_1 - t_2 path P_1 and an s_2 - t_1 path P_2 such that $(P_1 \cup P_2) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$ if and only if $(\mathcal{A}_{1,2} \cup \mathcal{A}_{2,1}) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$.

Proof: Suppose that there exist an s_1 - t_2 path P_1 and an s_2 - t_1 path P_2 such that $(P_1 \cup P_2) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$. Noting that $\mathcal{A}_{1,2} \subseteq P_1$ and $\mathcal{A}_{2,1} \subseteq P_2$, we have $(\mathcal{A}_{1,2} \cup \mathcal{A}_{2,1}) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$, which proves the necessity.

Now we prove the sufficiency. Without loss of generality, suppose all the s_1 - t_2 paths intersect $\mathcal{A} = \mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \{e_1, e_2, \dots, e_n\} = P \cap Q$ for some s_1 - t_1 path P and some s_2 - t_2 path Q , where the existence of P and Q is guaranteed by Theorem 3.7. Now take an arbitrary s_1 - t_2 path P_1 , and let $e_i \in P_1$ for some $1 \leq i \leq n$. One can prove that $e_j \in P_1$ for all $1 \leq j \leq n$. In fact, when $j < i$, $P_1[s_1, e_i] - P[\text{head}(e_i), t_1]$ is an s_1 - t_1 path and hence contains \mathcal{A} , which implies that e_j lies in P_1 for any $1 \leq j < i$. When $j > i$, then $Q[s_2, e_i] - P_1[\text{head}(e_i), t_2]$ is an s_2 - t_2 path and hence contains \mathcal{A} . Therefore $e_j \in P_1$ for any $i < j \leq n$. The above discussions show that $\mathcal{A} \subseteq P_1$. Since P_1 is chosen arbitrarily, one can have that \mathcal{A} is contained in all the s_1 - t_2 paths, which means $\mathcal{A} \subseteq \mathcal{A}_{1,2}$, and thus $\mathcal{A}_{1,2} \cap \mathcal{A} = \mathcal{A} \neq \emptyset$. Similarly, when all s_2 - t_1 paths intersect \mathcal{A} , we have $\mathcal{A}_{2,1} \cap \mathcal{A} = \mathcal{A} \neq \emptyset$. Therefore $(\mathcal{A}_{1,2} \cup \mathcal{A}_{2,1}) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = (\mathcal{A}_{2,1} \cup \mathcal{A}_{1,2}) \cap \mathcal{A} = \mathcal{A} \neq \emptyset$, and the sufficiency holds. \blacksquare

Now we give our main result.

Theorem 4.8: Let $\mathcal{N} = (V, E, \{s_1, t_1\}, \{s_2, t_2\})$ be a 2-pair unicast network such that $C(s_i, t_j) = 1$ for $i, j = 1, 2$. Then the following statements are equivalent:

- (1) \mathcal{N} is available.
- (2) \mathcal{N} contains one of the four networks depicted in Fig.1.
- (3) $(\mathcal{A}_{1,2} \cup \mathcal{A}_{2,1}) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$

Proof: The equivalency between (1) and (2) has already been obtained by Theorem 4.3. Also, we have shown that \mathcal{N} is available if and only if $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \emptyset$ or $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$ and there exist an s_1 - t_2 path P_1 and an s_2 - t_1 path P_2 such that $(P_1 \cup P_2) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$, which is equivalent to $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} = \emptyset$ or $\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2} \neq \emptyset$ and $(\mathcal{A}_{1,2} \cup \mathcal{A}_{2,1}) \cap (\mathcal{A}_{1,1} \cap \mathcal{A}_{2,2}) = \emptyset$ by Lemma 4.7. Thus (1) and (3) are equivalent. \blacksquare

Note that the 2-pair unicast problem just aims at supporting two unit flows. *It is adequate to assume the information edges, $S(i)$ and $T(i)$ to have unit capacities.* Under such an assumption, \mathcal{N} always satisfies $C(s_i, t_j) = 1$ for $i, j = 1, 2$. Thus, the solvability of a 2-pair unicast problem is completely determined by the relations of $\mathcal{A}_{1,1}$, $\mathcal{A}_{2,2}$, $\mathcal{A}_{1,2}$, and $\mathcal{A}_{2,1}$ of the underlying network.

V. CONCLUSIONS AND DISCUSSIONS

In this paper, we proposed a subnetwork decomposition/combination approach and decomposed a 2-pair network into four point-to-point subnetworks $\mathcal{N}_{i,j}$, for $i, j = 1, 2$. It showed that the solvability of a 2-pair unicast problem is completely determined by four link subsets, $\mathcal{A}_{1,1}$, $\mathcal{A}_{2,2}$, $\mathcal{A}_{1,2}$, and $\mathcal{A}_{2,1}$ of the underlying network. The structure of the 2-pair unicast networks was developed by analyzing the relations of the \mathcal{A} -sets. As a result, it deduced four specific simple available networks, such that any available 2-pair unicast network contains one copy of them and vice versa. Our results yielded an efficient algorithm to determine the solvability

of the 2-pair unicast problem and a new proof that nonlinear network coding is unnecessary for solving the 2-pair unicast problem.

According to [22], the \mathcal{A} -set of a point-to-point network is composed by the links with *capacity rank* 1. It is reasonable to conjecture that the rate region of a general multi-source multi-sink network is merely determined by the “important links,” i.e., the links with small capacity ranks. Moreover, it will be valuable to obtain an equation similar to (3) of Theorem 4.8 for the general k -pair unicast networks.

The four proposed underlying networks have the property that any available 2-pair unicast network contains one copy of them. From such a sense, we call them a *minimum available family under network coding* for the 2-pair unicast networks. To decide such minimum available family for 3-pair or k -pair unicast networks in general is still open.

We focused on directed acyclic 2-pair unicast networks in this paper. For the undirected networks, it is conjectured that network coding have no more advantages than fractional routing, which is known as the *undirected k -pair conjecture* [15]. To find out the *minimum available family under fractional routing* for undirected k -pair networks is also a more challenging topic.

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